THE LINE ANALOG OF RAMSEY NUMBERS*

BY

MEHDI BEHZAD AND HEYDAR RADJAVI

ABSTRACT

For positive integers r and s, $f'(r, s)$ is defined as the smallest positive integer p such that every connected (ordinary) graph of order p contains either r mutually adjacent lines or s mutually disjoint lines. It is found that $f'(r, s) = (r-1)(s-1) + 2$ unless $r = 2$ and $s \neq 1$, in which case $f'(2, s) = 3$.

By the Ramsey number $f(r,s)$ is meant the smallest positive integer p such that every graph (finite, undirected, with no loops or multiple lines) of order p contains either r mutually adjacent points or s mutually disjoint points. These numbers have been studied extensively by Erdös [1,2,3, and 4], Erdös and Szekeres [5], Greenwood and Gleason [6], and others, who have found various bounds for $f(r,s)$. The exact values of $f(r,s)$ are not, in general, known.

In this note we consider the line analog of this problem to which we give a complete solution. First we observe that in the definition, given above, of $f(r,s)$ the class of graphs can be restricted to that of connected ones except for the trivial cases in which $r = 2$. We now define, for positive integers r and s, $f'(r,s)$ as the smallest positive integer p such that every connected graph of order p contains either r mutually adjacent lines or s mutually disjoint lines. Our main result (Theorem 3) gives the exact value of $f'(r,s)$ for every r and s.

It is worth mentioning here that if the word "connected" is omitted in the above definition, then $f'(r,s)$ would not exist.

For convenience we introduce the symbol $\mathscr{A}(r,s)$ to denote the class of all connected graphs which have either r mutually adjacent lines or s mutually disjoint lines.

LEMMA 1. *Let $g(r,s)$ be the smallest positive integer p such that every tree of order p is in $\mathscr{A}(r,s)$. Then $g(r,s) = f'(r,s)$.*

Proof. It suffices to show that $f'(r,s) \leq g(r,s)$. By the definition of $f'(r,s)$ there exists a connected graph G of order $f'(r,s) - 1$ which is not in $\mathscr{A}(r,s)$. Let G_0 be a spanning tree of G (a subgraph of G which is a tree and contains all the points of G). But G_0 cannot belong to $\mathscr{A}(r,s)$, implying that $g(r,s) > f'(r,s) - 1$.

Received January 9, 1967.

* Definitions not given here can be found in [7, 8].

The above lemma permits us to restrict ourselves to trees in what follows.

LEMMA 2. *In any non-trivial tree there exist two adjacent points a and b, $\deg a = 1$, such that at most one point adjacent with b has degree greater than* 1.

Proof. Assuming that the lemma holds for all trees with less than p points, $p > 1$, we use induction as follows. Let v be any point of a tree G of order p with $\deg v = 1$, and let u be the point adjacent with v. If $\deg u \leq 2$, the lemma immediately follows. Assuming $\deg u > 2$, we observe that the graph G_0 obtained from G by the removal of v is a tree of order $p - 1$. The points a and b of G_0 whose existence has been hypothesised are also effective for G.

Before stating Theorem 1, we mention that a *star graph* of order p is a tree all of whose $p - 1$ lines are incident with one point.

THEOREM 1. *For $r > 2$, $s > 1$, we have $f'(r,s) > (r-1)(s-1)+1$.*

Proof. For each pair of positive integers r and s, $r > 2$, $s > 1$, we shall construct a tree of order $(r-1)(s-1)+1$ which is not in $\mathscr{A}(r,s)$. Let $G_1, G_2, \cdots$, and G_{s-1} be $s - 1$ copies of the star graph of order r. For each i, $i = 1, 2, \cdots, s-2$, "identify" a point of degree 1 of G_i with a point of degree 1 of G_{i+1}, to obtain a tree T of order $(r-1)(s-1)+1$. (See Fig. 1 for an illustration in which $r = 6$ and $s = 5$.)

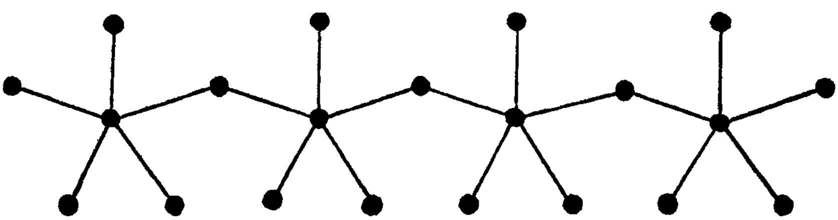

Fig. 1

Since T does not contain any point of degree r, in order to show that T is not in $\mathscr{A}(r,s)$, it suffices to observe that disjoint lines of T necessarily come from distinct G_i's.

In the proof of the next theorem we shall need the class of all trees of order $f(r,s) - 1$ which are not in $\mathscr{A}(r,s)$. For this reason we define $\mathscr{B}(r,s)$, $r > 2$, $s > 1$. For a fixed $r > 2$ let $\mathscr{B}(r,2)$ be the class consisting of the single star graph of order r. Having constructed $\mathscr{B}(r, s-1)$, we define $\mathscr{B}(r,s)$ as follows. Let T be any member of $\mathscr{B}(r, s-1)$. We "identify" any point of degree 1 of T with a point of degree 1 of the star graph of order r to obtain a tree G. The class $\mathscr{B}(r,s)$ is the set of all such trees as G. We note that every member of $\mathscr{B}(r,s)$ is of order $(r-1)(s-1) + 1$. As an illustration we mention that the trees given in Fig. 1 and Fig. 2 are, up to isomorphism, the only members of $\mathscr{B}(6,5)$.

THEOREM 2. *For $r > 2$, $s > 1$, we have $f'(r,s) = (r-1)(s-1)+2$.*

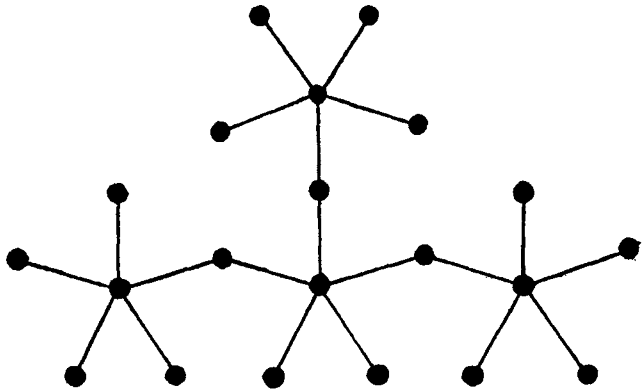

Fig. 2

Proof. By Theorem 1, $f'(r,2) > r$. Since every tree of order $r+1$ is either the star graph of order $r+1$ or else it contains two disjoint lines, we have $f'(r,2) = r+1$ for all $r > 2$. We note that for every $r > 2$ the star graph of order r is the only tree of this order which is not in $\mathscr{A}(r,2)$.

We now proceed by induction on s. For a fixed r, $r > 2$, assume that the members of $\mathscr{B}(r, s-1)$ are the only trees of order greater than or equal to $(r-1)(s-2)+1$ which are not in $\mathscr{A}(r, s-1)$. Now, by the definition of $f'(r,s)$ and by Lemma 1, there exists a tree of order $f'(r,s)-1$ not belonging to $\mathscr{A}(r,s)$. Suppose G is any such tree. Let a and b be the two points of G determined by Lemma 2. By the assumption on G, deg $b \leqq r-1$; hence the removal from G of all the lines incident with b will result in a tree G_0 of order P_0, $p_0 \geqq f'(r,s) - r$, together with some isolated points. It follows from Theorem 1 that $p_0 \geqq (r-1)(s-2)+1$. The induction hypothesis now implies that G_0 is in $\mathscr{B}(r, s-1)$. Hence $p_0 = f'(r,s) - r = (r-1)(s-2)+1$, from which it follows that

i) $f'(r,s) = (r-1)(s-1)+2$, and

ii) the degree of b in G is $r-1$, implying that G is in $\mathscr{B}(r,s)$.

Before stating the main result we observe that:

1) $f'(r,1) = 2$ for every r,

2) $f'(1,s) = 2$ for every s, and

3) $f'(2,s) = 3$ for all $s > 1$.

THEOREM 3. *For $r = 2$ and $s > 1$ we always have $f'(2,s) = 3$. For all other positive integers r and s the formula $f'(r,s) = (r-1)(s-1)+2$ holds.*

It is perhaps worth mentioning that, in contrast with the case of $f(r,s)$, the symmetricity in r and s of the function $f'(r,s)$, just established for almost all values of r and s, is not at all self-evident.

In conclusion we would like to thank Professors E. A, Nordhaus and B. M. Stewart of Michigan State University for pointing out an error in the original manuscript.

BIBLIOGRAPHY

1. P. Erdös, *Some remarks on the theory of graphs*, Bull. Amer. Math. Soc. **53** (1947), 292–294.

2. P. Erdös, *Remarks on a theorem of Ramsey*, Bull. Res. Counc. Israel **7F** (1957), 21–24.

3. P. Erdös, *Graph theory and probability* I, Canad. J. Math. **11** (1959), 34–38.

4. P. Erdos, *Graph theory and probablity* II, Canad. J. Math. **13** (1961), 346–352.

5. P. Erdös and G. Szekeres, *A combinatorial problem in geometry*, Composito Math. **2** (1935), 463–470.

6. R. E. Greenwood and A. M. Gleason, *Combinatorial relations and chromatic graphs*, Canad. J. Math. **7** (1955), 1–7.

7. F. Harary, *A seminar on graph theory*, Holt, Rinehart, and Winston, New York, (1967).

8. O. Ore, *Theory of graphs*, Amer. Math. Soc. Colloq. **38**, Providence (1962).

PAHLAVI UNIVERSITY,
SHIRAZ, IRAN